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Exploring Types of Open Sets in Neutrosophic Over Soft Topological Spaces with an Application to Organic Catalyst Selection

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ABSTRACT

Neutrosophic Over Soft Topological Spaces ($\mathcal{N}_s^o$–topological space) integrate the concepts of softness and neutrosophy , thereby providing a higher-level framework for handling uncertainty in topological structures ; this work extends this existing framework by introducingnew classes of generalized open sets, namely, $\mathcal{N}_s^o\alpha$-, $\mathcal{N}_s^o\beta$-, $\mathcal{N}_s^o$ semi- and $\mathcal{N}_s^o$ pre-open sets, and gives corresponding notions of continuity; many propositions concerning relationships of such types of functions are established, and next approach composition and behavior of such functions in various situations. In this paradigmatic case study, such an application of theoretical advancements turns to considering a numerical example, selecting the most suitable catalyst for a reaction of organic chemistry, revealing the possible role of decision making with the $\mathcal{N}_s^o$-topological space in complicating environments.

Key words: $\mathcal{N}_s^o\alpha$-open set, $\mathcal{N}_s^o\beta$ -open set, $\mathcal{N}_s^o$-semi open set, $\mathcal{N}_s^o$-pre open set, Generalized continuity, Catalyst selection.

INTRODUCTION

The introduction of fuzzy sets[28] by Zadeh in 1965 created the foundation for fuzzy set theory, permitting partial membership of elements and thereby inducing a revolution in decision-making and control systems. Zadeh also extended this idea in 1978 by establishing the frame of reference of possibility theory[29] as a more intuitive way to deal with uncertainty than probability theory. Then, in 1970, Bellman and Zadeh[3] laid the first applications of fuzzy logic in decision-making environments on these theoretical foundations, giving birth to a host of real-world applications in fields such as economics, engineering, and operations research. In 1995 Bustince and Burillo[5] contributed to uncertainty modeling through the study of interval-valued intuitionistic fuzzy sets, extending the expressive power of fuzzy sets through the inclusion of both Subsequently, in 1998, Atanassov and Shannon[1] expanded the concept of intuitionistic fuzzy logic by introducing logical operations and properties, thereby enlarging its theoretical horizon.

Molodtsov[18] introduced in 1999 the soft set theory notion as an innovative mathematical tool capable of fighting against vagueness in situations parameterized with applications where normal models could not be utilized. In parallel, Smarandache[23] introduced neutrosophic logic in the same year as a powerful generalization of fuzzy and intuitionistic fuzzy logic, distinguishing between degrees of truth, indeterminacy, and falsity, and thus allowing a more sophisticated approach to modeling uncertainty. Maji et al. in[15] in 2003 formalized a soft-set framework into a structured decision-making tool providing the groundwork for more integration with fuzzy and neutrosophic theories. Wang et al. [26] extended the entire neutrosophic theory through the introduction of single-valued neutrosophic sets in 2010, where the application of this theory could be better addressed in real-life decision support systems.

Ye[27] introduced some innovative single-valued neutrosophic correlation coefficients in 2013. Moreover, he proposed improved decision-making processes that would augment computational tools in dealing with uncertainty. In 2016, Smarandache[25] came up with oversets, undersets, and offsets to further develop neutrosophic theory. This expansion gave the theory more ways of structuring itself. Dhavaseelan *et al.*[10], in 2019, introduced the idea of neutrosophicα^m-continuity, adding to the study of generalized continuity in neutrosophic topologies. During the same year, RN Majeed and SA El-Sheikh[21] dealt with fuzzy orbit topological spaces offering some new applications in the field of materials science and engineering systems. Correlation measures associated with pythagorean neutrosophic sets were actually put forward by Jansi et al.[11] in 2019 giving better decision-making tools for complex uncertainty. Mehmood et al. developed the concept of neutrosophic softα-open sets in 2020,[17]marking another important step in soft set and neutrosophic topological theories.

In the same year, 2020, Saeed *et al.*[22] took on medical diagnosis through applications of multi-polar neutrosophic soft sets which reveal the capability of neutrosopholic models in healthcare settings. The same year, Christianto et al.[6] investigated some philosophical and scientific implications of neutrosophic logic to encourage its adoption in mainstream physical sciences. Smarandache and Pramanik[24] compiled a hefty edited book in 2020, which captures the most recent developments and future directions of neutrosophistic theory. Mallick and Pramanik [16] are also in line, proposing pentapartitioned neutrosophic sets which will accommodate much more assorted forms of uncertainty without extending them from classical partition.

Radha *et al.*,[20] in 2021 presented neutrosophic Pythagorean sets with dependent components which improved correlation coefficients and hence considerably boosted the evaluation of such-related similarity in uncertain datasets as well as strengthened decision-making systems. In 2021, Madhumathi and Nirmala Irudayam[13,14] introduced neutrosophic orbit topological spaces, and in 2022, orbit continuous mappings, these are new instruments that studies dynamical and parametric uncertainty via topological structures. In 2022, Atanassov established the intuitionistic fuzzy modal topological structure[2], which incorporated modal operators into fuzzy logic and paved the way for new possibilities in reasoning under graded necessity and possibility. Broumi[4] has enriched the conceptualization of generalized neutrosophic soft sets in terms of the better analytical formulation.

Kumaravel *et al.*[12] applied fuzzy and neutrosophic cognitive maps for disease diagnosis and COVID variant modeling by Murugesan et al.[19] This relevance is highlighted with respect to the real-world issues today. From 2024 to 2025, Devi and Parthiban[7,8,9] have incorporated neutrosophic approach over soft topological constructs in decision making problem.

This study focuses on the extension of Neutrosophic Over Soft Topological Spaces ($\mathcal{N}_{\mathfrak{s}}^{o}$-spaces) by introducing new generalized open sets, namely $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open, $\mathcal{N}_{\mathfrak{s}}^{o}\beta$ -open, $\mathcal{N}_{\mathfrak{s}}^{o}$ -semi open, and $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open sets. Various types of continuity corresponding to these open sets are investigated, including$\mathcal{N}_{\mathfrak{s}}^{o}$-continuous, $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$ -continuous,$\mathcal{N}_{\mathfrak{s}}^{o}\beta$ -continuous, semi-continuous, and pre-continuous functions. These enhancements provide a deeper understanding the structural behavior of $\mathcal{N}_{\mathfrak{s}}^{o}$-spaces. Additionally, a numerical application related to optimal catalyst selection is presented to demonstrate the effectiveness of the proposed framework. By systematically exploring the interrelationships among the newly defined open sets and their associated continuity types , this study establishes a comprehensive view of their mutual dependencies . Such an investigation reveals how each class of set and function interacts with the others , leading to a clearer understanding of the underlying topological framework . Nevertheless, This integrated perspective strengthens the theoretical foundations of neutrosophic over soft topological spaces and highlights previously unexplored structural properties . As a result , the study contributes to building a more robust and versatile mathematical structure for $\mathcal{N}_{\mathfrak{s}}^{-}$paces.

Preliminaries

This section provides the fundamental concepts and definitions necessary for understanding Neutrosophic Over Soft Set[9] and Neutrosophic Over Soft Topological Spaces[9].

Definition 2.1 [9] Let $\mathcal{H}$ be an non-empty set and be a set of parameter on .Then $\mathcal{N}_{\mathfrak{s}}^{o}$-set is defined by a set valued function

$$\lambda\mathcal{N}_{\mathfrak{s}}^{o}\colon \mathcal{E}\to\rho(\mathcal{H})$$

where $\rho(\mathcal{H})$ is an set of all $\mathcal{N}_{\mathfrak{s}}^{o}$-set on $\mathcal{H}.$ $\mathcal{N}_{\mathfrak{s}}^{o}$-set is an valued function from the set of parameter $\mathcal{E}$ on $\mathcal{H}$ is defined as

$$\mathcal{J} = (\lambda\mathcal{N}_{\mathfrak{s}}^{o}, \mathcal{E}) = \{(\mathrm{e}, \{\langle \mathrm{h}, \aleph\mathcal{J}(\mathrm{h}), \eth\mathcal{J}(\mathrm{h}), \Upsilon\mathcal{J}(\mathrm{h})\rangle\colon \mathrm{h} \in\mathcal{H}\})\colon \mathrm{e} \in\mathcal{E}\}$$

Definition 2.2 [9]

A $\mathcal{N}_{\mathfrak{s}}^{o}$-set ▷ = $\{e, \{\langle h, 0,0, \Omega\rangle\colon h\in\mathcal{H}\}\colon e\in\mathcal{E}\}$ is said to be a Null $\mathcal{N}_{\mathfrak{s}}^{o}$-set and ⍰$= \{e, \{\langle h, \Omega, \Omega, 0\rangle\colon h\in\mathcal{H}\}\colon e\in\mathcal{E}\}$ is said to be an universal $\mathcal{N}_{\mathfrak{s}}^{o}$-set.

Definition 2.3 [9] A neutrosophic over soft topology($\mathcal{N}_{\mathfrak{s}}^{o}$-topology) $\tau\mathcal{N}_{\mathfrak{s}}^{o}$ on
non-empty set such that
(i) ▷, ▶ $\in\tau\mathcal{N}_{\mathfrak{s}}^{o}$.
(ii) The union of an arbitrary collection $\tau\mathcal{N}_{\mathfrak{s}}^{o}$ is in $\tau\mathcal{N}_{\mathfrak{s}}^{o}$.
(iii) The finite intersection of subsets $\tau\mathcal{N}_{\mathfrak{s}}^{o}$ is in $\tau\mathcal{N}_{\mathfrak{s}}^{o}$.
Then $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ is called neutrosophic over soft topological space($\mathcal{N}_{\mathfrak{s}}^{o}$-topological space).An element of $\tau\mathcal{N}_{\mathfrak{s}}^{o}$ is called an neutrosophic over soft open set($\mathcal{N}_{\mathfrak{s}}^{o}$ –openset) and complement of $\tau\mathcal{N}\mathfrak{so}$ neutrosophic over soft closed set($\mathcal{N}_{\mathfrak{s}}^{o}$ –closedset).

Definition 2.4 [9]

An operators of $\mathcal{N}_{\mathfrak{s}}^{o}$-set $\mathcal{J}\in\tau\mathcal{N}_{\mathfrak{s}}^{o}$, then neutrosophic over soft topological closure and interior are $cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$ and $int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$ is defined as:
$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$ = {$\mathcal{G}\colon \mathcal{G}\ is\ \mathcal{N}_{\mathfrak{s}}^{o} - closed\ set\ in\ \mathcal{H}$and $\mathcal{J}\subseteq\mathcal{G}$}.
$int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}) = \{\mathcal{O}\colon\ \mathcal{O}\ is\ \mathcal{N}_{\mathfrak{s}}^{o}\ -open\ set\ in\ \mathcal{H}$ and $\mathcal{J}\supseteq\mathcal{O}$}.

Note

In this paper, the Neutrosophic Over Soft Set $\mathcal{J}$ is defined initially as

$$\mathcal{J} = (\lambda\mathcal{N}_{\mathfrak{s}}^{o}, \mathcal{E}),$$

in terms of elements $\mathrm{e} \in\mathcal{E}$ and corresponding maps involving $\mathrm{h} \in\mathcal{H}$. Yet, in the main work throughout, $\mathcal{J}$ is always represented as $\mathcal{J} = \{\langle \mathrm{h}, \aleph\mathcal{J}(\mathrm{h}), \eth\mathcal{J}(\mathrm{h}), \Upsilon\mathcal{J}(\mathrm{h})\rangle\colon \mathrm{h} \in\mathcal{H}\}$, emphasizing the basic relations within the framework.

Characterization of Open Set Types in Neutrosophic Over Soft Sets

Definition 3.1 Let $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ be a $\mathcal{N}_{\mathfrak{s}}^{-}$opological space and let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{o}$-set of $\mathcal{H}$
then $\mathcal{J}$ is said to be
i. $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open if $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})))$
ii. $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open if $\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})))$
iii. $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open if $\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$
iv. $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open if $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$

Proposition

i. Every $\mathcal{N}_{\mathfrak{s}}^{o}$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open.
ii. Every $\mathcal{N}_{\mathfrak{s}}^{o}$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open.

iii. Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open set is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open.
iv. Every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open set is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open.

Proof. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open set in a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space $(\mathcal{H},\, \tau\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}})$.

i. Since $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open, we have $\mathcal{J} = int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$. Now consider
$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))) = int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$
Hence, $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$, which implies that $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open.

ii. Since $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open, as above, we have $\mathcal{J} = int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$. Then
$cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))) = cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))) \supseteq \mathcal{J}$
since $cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) \supseteq \mathcal{J}$ and $int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})) \supseteq \mathcal{J}$ because $\mathcal{J}$ is open. Hence $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open.

iii. Since $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open, then $int\mathcal{N}\mathfrak{so}(\mathcal{J}) = \mathcal{J}$ and so $cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})) =$
$cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) \supseteq \mathcal{J}$, hence $\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$ which proves $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open.

iv. Similarly, since $\mathcal{J} = int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$, and $cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) \supseteq \mathcal{J}$, we have
$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})) \supseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) = \mathcal{J}$
Hence $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$, so $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open.
Therefore, every $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open set is also $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open, $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open, $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open and $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open.

Proposition 3.3

Let $(\mathcal{H},\, \mathcal{T}\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}})$ be a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space. Then
i. The union of any family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open sets is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open.
ii. The union of any family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open sets is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open.
iii. The union of any family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open sets is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open.
iv. The union of any family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open sets is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open.

Proof. Let $\{\mathcal{J}_\lambda\}_{\lambda\in\Lambda}$ be a family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open sets in a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space
$(\mathcal{H},\, \tau\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}})$.

i. By definition of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open set, for each $\lambda\in\Lambda$, we have:

$$\mathcal{J}_\lambda\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$$

Now consider the union $\mathcal{J} =_{\lambda\in\Lambda}\mathcal{J}_\lambda$.
To prove that $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})))$.

Since the interior and closure operators are monotone and preserve unions over open sets, we get:
$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) = int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\ _{\lambda\in\Lambda}\mathcal{J}_\lambda) =_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$
$cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})) = cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\,_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$

$$\subseteq_{\lambda\in\Lambda} cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$$ Applying the interior operator again:
$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))) \supseteq_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda))$ Now, since $\mathcal{J}_\lambda\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)))$ for each λ, we have: $_{\lambda\in\Lambda}\mathcal{J}_\lambda\subseteq_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda))) \subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})))$ Hence,

$$\mathcal{J}=_{\lambda\in\Lambda}\mathcal{J}_\lambda\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})))$$

Therefore, $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open.

ii. Let $\{\mathcal{J}_\lambda\}_{\lambda\in\Lambda}$ be a family of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open sets in a $\mathcal{N}_{\mathfrak{s}}$-opological space $(\mathcal{H},\, \tau\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}})$.
By definition of $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open sets, for each $\lambda\in\Lambda$, we have:

$$\mathcal{J}_\lambda\subseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$$

Let $\mathcal{J} =_{\lambda\in\Lambda}\mathcal{J}_\lambda$. We want to show:

$$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$$

First, since $\mathcal{J} =_{\lambda\in\Lambda}\mathcal{J}_\lambda$, and the closure operator is monotonic and preserves unions:

$$cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) = cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\ _{\lambda\in\Lambda}\mathcal{J}_\lambda) \supseteq_{\lambda\in\Lambda} cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$$

Applying the interior operator (which is also monotonic):

$$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) \supseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\,_{\lambda\in\Lambda} cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$$

$\supseteq_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda)$ Now apply closure again:
$cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\ int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}) \supseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\,_{\lambda\in\Lambda} int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda) \supseteq_{\lambda\in\Lambda} cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda))$
Therefore:

$$_{\lambda\in\Lambda}\mathcal{J}_\lambda\subseteq_{\lambda\in\Lambda} cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}_\lambda))$$

$\subseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$ Hence,
$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J}))$
Therefore, $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open. For iii and iv It is obviously true.

Proposition 3.4

Let $(\mathcal{H},\, \tau\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}})$ be $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space then
i. $\triangleright$ and $\blacktriangleright$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open sets.
ii. $\triangleright$ and $\blacktriangleright$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\beta$-open sets.
iii. $\triangleright$ and $\blacktriangleright$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-semi open sets.
iv. $\triangleright$ and $\blacktriangleright$ are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-pre open sets.

Proof. i. By definition of a $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-topological space, the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ null set $\triangleright$ and $\blacktriangleright$ the $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$ universal set are elements of $\tau\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$. Hence, they are $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}$-open sets.

We now verify whether these sets satisfy the condition for being $\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}\alpha$-open. That is, we need to check whether: $\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\mathcal{J})$
for $\mathcal{J} = \triangleright$ and $\mathcal{J} = \blacktriangleright$.
Case 1: Let $\mathcal{J} = \triangleright$.
Since $\triangleright$ is open, we have:
$int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\triangleright) = \triangleright \Rightarrow cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(int\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\triangleright)) = cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\triangleright) \Rightarrow cl\mathcal{N}_{\mathfrak{s}}^{\mathfrak{o}}(\triangleright) = \triangleright$
(because closure of the empty set is empty in any

topological space)

$\Rightarrow int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\vartriangleright)) = int\mathcal{N}_{\mathfrak{s}}^{o}(\vartriangleright) = \vartriangleright$

Hence, $\vartriangleright \subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\vartriangleright)))$.

Therefore, $\vartriangleright$ is $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open.

Case 2: Let $\mathcal{J} = \blacktriangleright$.

Since is open, we have:

$int\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright) = \Rightarrow cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright)) = cl\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright) =$

$\Rightarrow int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright)) = int\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright) =$

Hence, $\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\blacktriangleright)))$.

Therefore, is $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open.

ii, iii and iv are obviously true.

Proposition 3.5 Every $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open.

Proof. Let $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ be a $\mathcal{N}_{\mathfrak{s}}^{o}$-topological space and let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set.

By definition of $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set, we have:

$\mathcal{J}\subseteq int\mathcal{N}\mathfrak{s}o\,cl\mathcal{N}\mathfrak{s}o\,int\mathcal{N}\mathfrak{s}o(\mathcal{J})$ Let us denote $A = int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$.

$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(A)$ ...(3.1)

Since $A\subseteq\mathcal{J}$, it follows that:

$cl\mathcal{N}_{\mathfrak{s}}^{o}(A) \subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$

Now apply the interior operator on both sides:

$int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(A)) \subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$ ...(3.2)

From 3.1 and 3.2

$$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$$

This is precisely the definition of a $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open set. Hence, every $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open.

Remark 3.6

Converse of proposition3.5 is need not to be true which is proven by the example 3.7

Example 3.7

Let $\mathcal{H} = \{x_1, x_2, x_3\}$ and define a $\mathcal{N}_{\mathfrak{s}}^{o}$-topology on by: $\tau\mathcal{N}_{\mathfrak{s}}^{o} = \{\vartriangleright, \blacktriangleright, \mathcal{V}_1, \mathcal{V}_2\}$, where, $\mathcal{V}_1 = \{\langle x_1, 1.2,0.4,0.3\rangle, \langle x_2, 1.3,0.2,0.4\rangle, \langle x_3, 1.3,0.2,0.1\rangle\}$,

$\mathcal{V}_2 = \{\langle x_1, 1.1,0.3,0.5\rangle, \langle x_2, 1.2,0.2,0.5\rangle, \langle x_3, 1.2,0.1,0.2\rangle\}$.

Define the neutrosophic over soft set: $\mathcal{K} = \{\langle x_1, 1.1,0.3,0.5\rangle, \langle x_2, 1.2,0.1,0.5\rangle, \langle x_3, 1.2,0.1,0.2\rangle\}$

$(\tau\mathcal{N}_{\mathfrak{s}}^{o})^C = \{\vartriangleright, \blacktriangleright, \mathcal{V}_1, \mathcal{V}_2\}$, where $\mathcal{V}_1 = \{\langle x_1, 0.3,1.1,1.2\rangle, \langle x2, 0.4,1.3,1.3\rangle, \langle x_3, 0.1,1.3,1.3\rangle\}$

$\mathcal{V}_2 = \{\langle x_1, 0.5,1.2,1.1\rangle, \langle x_2, 0.4,1.3,1.2\rangle, \langle x_{3,} 0.2,1.4,1.2\rangle\}$

$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K}) =$

$int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})) = \mathcal{V}_1 \quad (3.3)$

$\mathcal{K}\subset\mathcal{V}_1 \quad (3.4)$

From 3.3 and 3.4

$\mathcal{K}\subset int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})) \qquad \therefore\mathcal{K}$ is $\mathcal{N}_{\mathfrak{s}}^{o} -$ preopen

$int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K}) = \vartriangleright$

$cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})) = \mathcal{V}1$

$int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K}))) = \vartriangleright$

$int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K}))) = \vartriangleright$......(3.5)

$\mathcal{K}\nsubseteq\vartriangleright$ ....(3.6)

From (3.5) and (3.6)

$\mathcal{K}\nsubseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})))$

$\therefore\mathcal{K}$ is not $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$ – open

Proposition 3.8

Every $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open.

Proof. Let $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ be a neutrosophic over soft topological space. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set. By definition of $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set, we have:

$$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$$

Now let us denote $\mathcal{K} = int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$, which is open set since the interior of any set in a topology is open by definition.

So we have

$$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})$$

Since $\mathcal{K}$ is open, its closure $cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})$ is a superset of $\mathcal{K}$ and thus contains all points that are limit points or in $\mathcal{K}$.

The interior of the closure, $int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})$, is therefore an open set in $\tau\mathcal{N}_{\mathfrak{s}}^{o}$ that contains $\mathcal{J}$.

Now, recall the definition of a $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open set: A set $\mathcal{J}$ is said to be semi open if

$$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$$

But from the assumption that $\mathcal{J}$ is $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open, we already have:

$$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K}) \Rightarrow \mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{K})$$

(because any set is contained in its interior of the closure implies it's also in the closure itself).

Since $\mathcal{K} = int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$, it follows that:

$$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$$

which is the definition of a $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open set.

Hence, every $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set is indeed $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open.

Example 3.9 Let $\mathcal{H} = \{x_1, x_2, x_3\}$ and define a $\mathcal{N}_{\mathfrak{s}}^{-}$opology on $\mathcal{H}$ by:

$$\tau\mathcal{N}_{\mathfrak{s}}^{o} = \{\vartriangleright, \blacktriangleright, \mathcal{V}_1, \mathcal{V}_2\},$$

where

$\mathcal{V}_1 = \{\langle x_1, 1.1,0.4,0.5, \langle x_2, 1.2,0.3,0.3\rangle, \langle x_3, 1.3,0.2,0.4\rangle\}$,

$\mathcal{V}_2 = \{\langle x_1, 1.2,0.5,0.6, \langle x_2, 1.3,0.4,0.4\rangle, \langle x_3, 1.4,0.3,0.3\rangle\}$.

Let the neutrosophic over soft set be:

$$\mathcal{M} = \{\langle x_1, 1.2,0.5,0.6\rangle, \langle x_2, 1.3,0.4,0.4\rangle, \langle x_3, 1.4,0.3,0.3\rangle\}.$$

Clearly,

$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{M}) = \mathcal{V}_2, \quad int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{M})) = \mathcal{V}_1.$

So,

$\mathcal{M}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{M})) = \mathcal{V}_1 \Rightarrow \mathcal{M}is\mathcal{N}_{\mathfrak{s}}^{o} -$ semiopen.

However,

$int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{M}))) = \rhd$, $\mathcal{M}\nsubseteq\rhd$.

$\therefore\mathcal{M}$ is $\mathcal{N}_{\mathfrak{s}}^{o}$ –semi open but not $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$ – open.

Proposition 3.10 Every $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open.

Proof. Let $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ be a $\mathcal{N}_{\mathfrak{s}}^{o}$-topological space, and let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set.

By definition of $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-openness, we have:

$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$

Now observe that:

$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}) \supseteq\mathcal{J}\Rightarrow int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}) \subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$

Then,

$cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})) \subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})))$

Since

$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))),$

and from the above inclusion, we conclude that:

$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}))$

This is precisely the condition for $\mathcal{J}$ to be $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open.

Hence, every $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-semi open.

Example 3.11 Let $\mathcal{H} = \{x_1, x_2, x_3\}$ and define a $\mathcal{N}_{\mathfrak{s}}$-opology on $\mathcal{H}$ as:

$\tau\mathcal{N}_{\mathfrak{s}}^{o} = \{\rhd, \blacktriangleright, \mathcal{W}_1, \mathcal{W}_2\},$

where

$\mathcal{W}_1 = \{\langle x_1, 1.2,0.3,0.4\rangle, \langle x_2, 1.3,0.2,0.3\rangle, \langle x_3, 1.2,0.4,0.2\rangle\},$

$\mathcal{W}_2 = \{\langle x_1, 1.1,0.4,0.5\rangle, \langle x_2, 1.2,0.3,0.4\rangle, \langle x_3, 1.3,0.2,0.3\rangle\}.$

Define the neutrosophic over soft set:

$\mathcal{A} = \{\langle x_1, 1.1,0.4,0.5\rangle, \langle x_2, 1.2,0.3,0.4\rangle, \langle x_3, 1.3,0.2,0.3\rangle\}.$

We observe:

$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A}) = \mathcal{W}_{2,} \quad int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A})) = \mathcal{W}_1.$

Thus,

$\mathcal{A}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A})) = \mathcal{W}_1,$

$\therefore\mathcal{A}$is$\mathcal{N}_{\mathfrak{s}}^{o} -$ semiopen.

But now consider:

$int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A}) = \rhd , \quad cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A})) =$ ⍰.

Since

$\mathcal{A}\nsubseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A})) =$ ⍰,

we conclude that

$\mathcal{A}$ is not $\mathcal{N}_{\mathfrak{s}}^{o}\beta -$ open.

Proposition 3.12 Every $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set is $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open.

Proof. Let $(\mathcal{H}, \tau\mathcal{N}_{\mathfrak{s}}^{o})$ be a neutrosophic over soft topological space. Let $\mathcal{J}$ be a $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set in this space. By definition of a $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set, we have:

$\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}).$

On the other hand, a $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open set $\mathcal{J}$ is defined as:

$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}).$

Since the interior operator is monotonic, meaning that for any sets $\mathcal{A}\subseteq\mathcal{B}$, we have

$int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{A}) \subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B}),$

and since closure is extensive, i.e., $\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$, applying closure and then interior

again will only shrink or retain the set:

$int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}) \subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}).$

So if $\mathcal{J}\subseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J})$, then it follows that

$\mathcal{J}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{J}),$

which means that $\mathcal{J}$ is also $\mathcal{N}_{\mathfrak{s}}^{o}$-pre open.

Example 3.13 Let $\mathcal{H} = \{x_1, x_2, x_3\}$ be a universe, and define a $\mathcal{N}_{\mathfrak{s}}^{o}$-topology by:

$\tau\mathcal{N}_{\mathfrak{s}}^{o} = \{\rhd, \blacktriangleright, \mathcal{V}_1, \mathcal{V}_2\},$

where

$\mathcal{V}_1 = \{\langle x_1, 1.2,0.3,0.4\rangle, \langle x_2, 1.1,0.4,0.3\rangle, \langle x_3, 1.3,0.2,0.3\rangle\},$

$\mathcal{V}_2 = \{\langle x_1, 1.1,0.4,0.4\rangle, \langle x_2, 1.2,0.3,0.4\rangle, \langle x_3, 1.2,0.2,0.3\rangle\}.$

Now define the neutrosophic over soft set:

$\mathcal{B} = \{\langle x_1, 1.1,0.4,0.4\rangle, \langle x_2, 1.2,0.3,0.4\rangle, \langle x_3, 1.2,0.2,0.3\rangle\}.$

We observe:

$cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B}) = \mathcal{V}_2, \quad int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B})) = \mathcal{V}_1.$

So,

$\mathcal{B}\subseteq int\mathcal{N}_{\mathfrak{s}}^{o}(cl\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B})), \quad \therefore\mathcal{B}$ is $\mathcal{N}_{\mathfrak{s}}^{o} -$ preopen.

Now check:

$int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B}) = \rhd, cl\mathcal{N}_{\mathfrak{s}}^{o}(int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B})) =$ ⍰.

Hence,

$\mathcal{B}\nsubseteq cl\mathcal{N}_{\mathfrak{s}}^{o}int\mathcal{N}_{\mathfrak{s}}^{o}(\mathcal{B}), \quad \therefore\mathcal{B}$ is not $\mathcal{N}_{\mathfrak{s}}^{o}\beta -$ open.

Definition 3.14 Let $(\mathcal{H}_1, \tau\mathcal{N}_{\mathfrak{s}}^{o}1)$ and $(\mathcal{H}_2, \tau\mathcal{N}_{\mathfrak{s}}^{o}2)$ be two $\mathcal{N}_{\mathfrak{s}}^{o}$-topological spaces. A

mapping $f: \mathcal{H}_1 \to \mathcal{H}_2$ is said to be:

i. $\mathcal{N}_{\mathfrak{s}}^{o}$-continuous Function if the pre-image of every $\mathcal{N}_{\mathfrak{s}}^{o}$-open set in $(\mathcal{H}_2, \tau\mathcal{N}_{\mathfrak{s}}^{o}2)$ is a $\mathcal{N}_{\mathfrak{s}}^{o}$-open set in $(\mathcal{H}_1, \tau\mathcal{N}_{\mathfrak{s}}^{o}1)$, i.e., $\forall\mathcal{O}\in\tau\mathcal{N}_{\mathfrak{s}}^{o}2$, $f^{-1}(\mathcal{O}) \in\tau\mathcal{N}_{\mathfrak{s}}^{o}1$.

ii. $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-continuous if the pre-image of every $\mathcal{N}_{\mathfrak{s}}^{o}$-open set in $(\mathcal{H}_2, \tau\mathcal{N}_{\mathfrak{s}}^{o}2)$ is

a $\mathcal{N}_{\mathfrak{s}}^{o}\alpha$-open set in $(\mathcal{H}_1, \tau\mathcal{N}_{\mathfrak{s}}^{o}1)$, i.e., $\forall\mathcal{A}\in\tau\mathcal{N}_{\mathfrak{s}}^{o}1$, $f^{-1}(\mathcal{A}) \in\tau\mathcal{N}_{\mathfrak{s}}^{o}2$.

iii. $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-continuous if the pre-image of every

$\mathcal{N}_{\mathfrak{s}}^{o}$-open set in $(\mathcal{H}_2, \tau\mathcal{N}_{\mathfrak{s}}^{o}2)$ is

a $\mathcal{N}_{\mathfrak{s}}^{o}\beta$-open set in $(\mathcal{H}1, \tau\mathcal{N}_{\mathfrak{s}}^{o}1)$, i.e., $\forall\mathcal{B}\in\tau\mathcal{N}_{\mathfrak{s}}^{o}1$, $f^{-1}(\mathcal{B}) \in\tau\mathcal{N}_{\mathfrak{s}}^{o}2$.

iv. $\mathcal{N}_{\mathfrak{s}}^{o}$-semi continuous if the pre-image of every $\mathcal{N}_{\mathfrak{s}}$-

emi open set in $(\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is a $\mathcal{N}_s^{o}$-semi open set in $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1)$, i.e.,$\forall\mathcal{S}\in\tau\mathcal{N}_s^{o}1$, $f^{-1}(\mathcal{S}) \in\tau\mathcal{N}_s^{o}2$.

v. $\mathcal{N}_s^{o}$-pre continuous if the pre-image of every $\mathcal{N}_s^{o}$-pre open set in
$(\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is a $\mathcal{N}_s^{o}$-pre open set in $\mathcal{H}_1$, i.e.,$\forall\mathcal{P}\in\tau\mathcal{N}_s^{o}1$,$f^{-1}(\mathcal{P}) \in\tau\mathcal{N}_s^{o}2$.

Proposition 3.15

i. Every $\mathcal{N}_s^{o}$-continuous function is $\mathcal{N}_s^{o}\alpha$-continuous.
ii. Every $\mathcal{N}_s^{o}$-continuous function is$\mathcal{N}_s^{o}\beta$-continuous.
iii. Every $\mathcal{N}_s^{o}$-continuous function is $\mathcal{N}_s^{o}$-pre continuous.
iv. Every $\mathcal{N}_s^{o}$-continuous function is $\mathcal{N}_s^{o}$-semi continuous.

Proof. Let f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o})$ be a $\mathcal{N}_s^{-}$ ontinuous function.
This means for every $\mathcal{N}_s^{o}$-open set $\mathcal{O}$ in $\mathcal{H}_2$, the preimage $f^{-1}(\mathcal{O})$ is a $\mathcal{N}_s^{o}$-open
set in $\mathcal{H}_1$.

i.$\mathcal{N}_s^{o}\boldsymbol{\alpha}$-continuity:
Let $\mathcal{A}$ be a $\mathcal{N}_s^{o}\alpha$-open set in $\mathcal{H}_2$, i.e.,
$\mathcal{A}\subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}(\mathcal{A})$.
Then
$f^{-1}(\mathcal{A}) \subseteq f^{-1}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}(\mathcal{A})$.
Using continuity of f, inverse image distributes over interior and closure:
$f^{-1}(\mathcal{A}) \subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}(f^{-1}(\mathcal{A}))$,
which shows that $f^{-1}(\mathcal{A})$ is $\mathcal{N}_s^{o}\alpha$-open. Hence, f is $\mathcal{N}_s^{o}\alpha$-continuous.

ii.$\mathcal{N}_s^{o}\beta$-continuity:
Let $\mathcal{B}$ be a $\mathcal{N}_s^{o}\beta$-open set in $\mathcal{H}_2$, i.e.,

$$\mathcal{B}\subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{B}).$$

Then
$f^{-1}(\mathcal{B}) \subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{B}))$,
which implies f is $\mathcal{N}_s^{o}\beta$-continuous.

iii.$\mathcal{N}_s^{o}$-pre continuity:
Let $\mathcal{P}$ be $\mathcal{N}_s^{o}$-pre open in $\mathcal{H}_2$, i.e.,
$\mathcal{P}\subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{P})$.
Then
$f^{-1}(\mathcal{P}) \subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{P}))$,
showing that $f^{-1}(\mathcal{P})$ is $\mathcal{N}_s^{o}$-pre open and hence f is $\mathcal{N}_s^{o}$-pre continuous.

iv.$\mathcal{N}_s^{o}$-semi continuity:
Let $\mathcal{S}$ be a $\mathcal{N}_s^{o}$-semi open set in $\mathcal{H}_2$, i.e.,
$\mathcal{S}\subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}(\mathcal{S})$.
Then
$f^{-1}(\mathcal{S}) \subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}(f^{-1}(\mathcal{S}))$,
so f is $\mathcal{N}_s^{o}$-semi continuous.
Thus, f is $\mathcal{N}_s^{o}\alpha$-, β-, pre-, and semi-continuous.

Proposition 3.16

Every $\mathcal{N}_s^{o}\alpha$-continuous function is $\mathcal{N}_s^{o}\beta$-continuous.

Proof. Let f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o})$ be a $\mathcal{N}_s^{-}$ ontinuous function. Let $\mathcal{A}$ be a
$\mathcal{N}_s^{o}$-open set in $\mathcal{H}_2$.
Then by definition of $\mathcal{N}\mathfrak{so}\beta$-open set, we have:
$\mathcal{A}\subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{A})$,
i.e.,
$\mathcal{A}\subseteq\{\mathcal{G}: \mathcal{G}$ is $\mathcal{N}_s^{o}$ – closed in $\mathcal{H}_2\}$ and $int\mathcal{N}_s^{o}(cl\mathcal{N}_s^{o}(\mathcal{A}))\subseteq\mathcal{G}$.
Take the preimage of both sides:

$f^{-1}(\mathcal{A}) \subseteq f^{-1}cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{A})$.
Using the assumption that f is $\mathcal{N}\mathfrak{so}\alpha$-continuous, and the fact that preimage commutes
with interior and closure, we get:
$f^{-1}(\mathcal{A}) \subseteq cl\mathcal{N}_s^{o}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{A}))$,
that is,
$f^{-1}(\mathcal{A}) \subseteq\{\mathcal{G}$: $\mathcal{G}$ is $\mathcal{N}_s^{o}$ – closed in $\mathcal{H}_1\}$ and $int\mathcal{N}_s^{o}(cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{A}))) \subseteq\mathcal{G}$,
which means $f^{-1}(\mathcal{A})$ is a $\mathcal{N}_s^{o}\beta$-open set in 1.
Hence, f is $\mathcal{N}_s^{o}\beta$-continuous.

Proposition 3.17

Every $\mathcal{N}_s^{o}\beta$-continuous function is $\mathcal{N}_s^{o}$-pre continuous.

Proof. Let f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o})$ be a $\mathcal{N}_s^{-}$ ontinuous function.
Let $\mathcal{P}$ be a $\mathcal{N}_s^{o}$-pre open set in $\mathcal{H}2$.
Then by the definition of $\mathcal{N}_s^{o}$-pre open set, we have:
$\mathcal{P}\subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{P})$,
i.e., $\mathcal{P}\subseteq\{\mathcal{O}$: $\mathcal{O}$ is $\mathcal{N}_s^{o}$—open in $\mathcal{H}_2\}$ and $\mathcal{O}\subseteq cl\mathcal{N}_s^{o}(\mathcal{P})$.
Now consider the preimage of both sides:
$f^{-1}(\mathcal{P}) \subseteq f^{-1}int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(\mathcal{P})$.
Since f is $\mathcal{N}_s^{o}\beta$-continuous, and since $\mathcal{P}$ is a subset of a composition involving a
$\mathcal{N}_s^{o}\beta$-open set, we can apply:
$f^{-1}(\mathcal{P}) \subseteq int\mathcal{N}_s^{o}cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{P}))$,
i.e., $f^{-1}(\mathcal{P}) \subseteq\{\mathcal{O}$: $\mathcal{O}$ is $\mathcal{N}_s^{o}$ – open in $\mathcal{H}_1\}$ and $\mathcal{O}\subseteq cl\mathcal{N}_s^{o}(f^{-1}(\mathcal{P}))$.
Hence, $f^{-1}(\mathcal{P})$ is $\mathcal{N}_s^{o}$-pre open in $\mathcal{H}_1$, which shows that f is $\mathcal{N}_s^{o}$-pre continuous.

Proposition 3.18

The composition of two $\mathcal{N}_s^{o}$-continuous functions is $\mathcal{N}_s^{o}$-continuous.

Proof. Let $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1)$, $(\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$, and $(\mathcal{H}_3, \tau\mathcal{N}_s^{o}3)$ be $\mathcal{N}_s^{o}$-topological spaces.
Let f: $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ and g: $\mathcal{H}_2 \rightarrow \mathcal{H}_3$ be two $\mathcal{N}_s^{-}$ ontinuous functions.
We must show that the composition $g\circ f$: $\mathcal{H}_1 \rightarrow \mathcal{H}_3$ is also $\mathcal{N}_s^{o}$-continuous.

Let $\mathcal{O}$ be any $\mathcal{N}_s^{o}$-open set in $\mathcal{H}_3$, i.e., $\mathcal{O}\in\tau\mathcal{N}_s^{o}3$.
Since g is $\mathcal{N}_s^{o}$-continuous, we have
$g^{-1}(\mathcal{O}) \in\tau\mathcal{N}_s^{o}2$.
Since f is $\mathcal{N}_s^{o}$-continuous, the preimage under f of this set is also $\mathcal{N}_s^{o}$-open in $\mathcal{H}_1$,
i.e.,
$f^{-1}(g^{-1}(\mathcal{O})) = (g\circ f)^{-1}(\mathcal{O}) \in\tau\mathcal{N}_s^{o}1$.
Therefore, the preimage of every $\mathcal{N}_s^{o}$-open set in $\mathcal{H}_3$ under $g\circ f$ is $\mathcal{N}_s^{o}$-open in $\mathcal{H}_1$.
Hence, $g\circ f$ is $\mathcal{N}_s^{o}$-continuous.

Proposition 3.19

The composition of two $\mathcal{N}_s^{o}\alpha$-continuous (respectively, β-, semi-, or pre-continuous) functions is again $\mathcal{N}_s^{o}\alpha$-continuous (respectively, β-, semi-, or pre-continuous).

Proof. Let $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1)$, $(\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$, and $(\mathcal{H}_3, \tau\mathcal{N}_s^{o}3)$ be $\mathcal{N}_s^{o}$-topological spaces.
Let f: $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ and g: $\mathcal{H}_2 \rightarrow \mathcal{H}_3$ be both $\mathcal{N}_s^{o}\alpha$-continuous (respectively, β-, semi-, or pre-continuous).
We aim to show that the composition $g\circ f$: $\mathcal{H}_1 \rightarrow \mathcal{H}_3$ is also $\mathcal{N}_s^{o}\alpha$-continuous
(respectively, β-, semi-, or pre-continuous).
Let $\mathcal{U}$ be a $\mathcal{N}_s^{o}\alpha$-open (respectively, β-, semi-, or pre-open) set in $\mathcal{H}_3$.
Since g is $\mathcal{N}_s^{o}\alpha$-continuous (respectively, β-, semi-, or pre-continuous), we have $g^{-1}(\mathcal{U}) \in\tau\mathcal{N}_s^{o}$
Since f is $\mathcal{N}_s^{o}\alpha$-continuous (respectively, β-, semi-, or pre-continuous), we have
$f^{-1}(g^{-1}(\mathcal{U})) = (g\circ f)^{-1}(\mathcal{U}) \in\tau\mathcal{N}_s^{o}$
Hence, the composition $g\circ f$ is $\mathcal{N}_s^{o}\alpha$ -continuous (respectively, β -, semi-, or pre-continuous).
Proposition 3.20 If f is a constant function between any two $\mathcal{N}_s^{o}$-topological spaces,
then f is $\mathcal{N}_s^{o}$-continuous.
Proof. Let $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1)$ and $(\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ be two $\mathcal{N}_s^{o}$-topological spaces, and let
f: $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a constant function.
Then, there exists an element, a$\in\mathcal{H}_2$ such that f(h) =a, for all h $\in\mathcal{H}1$.
Now, let $\mathcal{O}\in\tau\mathcal{N}_s^{o}2$ be any $\mathcal{N}_s^{o}$-open set in $\mathcal{H}_2$.
We consider two cases:
i. If a$\notin\mathcal{O}$, then $f^{-1}(\mathcal{O}) =\triangleright$, which is $\mathcal{N}_s^{o}$-open in $\mathcal{H}_1$.
ii. If a$\in\mathcal{O}$, then $f^{-1}(\mathcal{O}) = \mathcal{H}_1$, which is also $\mathcal{N}_s^{o}$-open in $\mathcal{H}_1$.
In both cases, $f^{-1}(\mathcal{O}) \in\tau\mathcal{N}_s^{o}1$, so f is $\mathcal{N}_s^{o}$-continuous.
Proposition 3.21 A bijective mapping f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is a
$\mathcal{N}_s^{o}$-homeomorphism if both f and f^{-1} are $\mathcal{N}_s^{-}$ontinuous.
Proof. Let f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ be a bijective mapping.
Assume f is $\mathcal{N}_s^{o}$-continuous. That is, for every $\mathcal{O}\in\tau\mathcal{N}_s^{o}2$, we have:
$f^{-1}(\mathcal{O}) \in\tau\mathcal{N}_s^{o}1$.
Also, assume f^{-1}: $\mathcal{H}_2 \rightarrow \mathcal{H}_1$ is $\mathcal{N}_s^{o}$-continuous. Then, for every $\mathcal{U}\in\tau\mathcal{N}_s^{o}1$, we have:
$f(f^{-1}(\mathcal{U})) = \mathcal{U}\Rightarrow f^{-1}(\mathcal{U}) \in\tau\mathcal{N}_s^{o}2$.
Since f is a bijection and both f and f^{-1} are $\mathcal{N}_s^{o}$-continuous, the mapping f
establishes a one-to-one, onto correspondence between the $\mathcal{N}_s^{o}$-open sets of $\mathcal{H}_1$ and $\mathcal{H}_2$.
Hence, f is a $\mathcal{N}_s^{o}$-homeomorphism.

Proposition 3.22

i. A bijective mapping f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is said to be a
$\mathcal{N}_s^{o}\alpha$-homeomorphism if both f and f^{-1} are $\mathcal{N}_s^{-}$ontinuous,
ii. A bijective mapping f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is said to be a
$\mathcal{N}_s^{o}\beta$-homeomorphism if both f and f^{-1} are $\mathcal{N}_s^{-}$ontinuous,
iii. A bijective mapping f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is said to be a $\mathcal{N}_s^{o}$-semi
homeomorphism if both f and f^{-1} are $\mathcal{N}_s^{o}$-semi continuous,
iv. A bijective mapping f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ is said to be a $\mathcal{N}_s^{o}$-pre
homeomorphism if both f and f^{-1} are $\mathcal{N}_s^{o}$-pre continuous.
Proof. i. Let f: $(\mathcal{H}_1, \tau\mathcal{N}_s^{o}1) \rightarrow (\mathcal{H}_2, \tau\mathcal{N}_s^{o}2)$ be a bijective function.
We consider case (i) for $\mathcal{N}_s^{o}\alpha$-continuity (other cases follow similarly):
Suppose f and f^{-1} are both $\mathcal{N}_s^{o}\alpha$-continuous. Then for any $\mathcal{O}_2\in\tau\mathcal{N}_s^{o}2$, we have:
$f^{-1}(\mathcal{O}_2) \in\tau\mathcal{N}_s^{o}1$,
where $\tau\mathcal{N}_s^{o}1$ denotes the collection of all $\mathcal{N}_s^{o}$-open sets in $\mathcal{H}_1$.
Similarly, for any $\mathcal{O}_1\in\tau\mathcal{N}_s^{o}1$,
$f(\mathcal{O}_1) \in\tau\mathcal{N}_s^{o}2$.
Hence, f establishes a one-to-one correspondence between $\mathcal{N}_s^{o}\alpha$-open sets of $\mathcal{H}_1$ and $\mathcal{H}_2$. Therefore, f is a $\mathcal{N}_s^{o}\alpha$-homeomorphism.

The proofs for ii, iii, and iv follow similarly by replacing α-open sets with β-, semi-, and pre-open sets respectively.

Numerical Application: Selection of Optimal Catalyst for Organic Reaction Using Neutrosophic Over Soft Topological Spaces

Choosing the best catalyst is really important in organic synthesis of large-volume industrial reactions with many other factors in play, like reaction speed, yield, cost, recycling ability, and environmental consideration. Catalysts have an effect not just on the conversion efficiency of raw materials into the desired product, but also on the doability of the process in economic terms, and, of course, its sustainability. However, in real-world laboratory and industrial environments, data with concern to the performance of a catalyst mostly comes as imprecise and incomplete, and then subjected to human interpretation. Expert chemists may point to different directions of efficiency and long-term usability; there can be uncertainty with experimental conditions; and the possible behavior of a catalyst can differ even under slightly changed conditions.

It is known that conventional mathematical models and classical logic based frameworks fail to address this inherent uncertainty and vagueness. Hence, interfacing neutrosophic logic, which constitutes truthiness, falsity, and indeterminate information, with soft topological structures is a robust instrument for making decisions. Neutrosophic over soft topological spaces give a flexibility and intelligent way of evaluating alternatives under imprecise or uncertain conditions since such spaces allow not only binary membership but also partial and ambiguous truths associated with performance parameters. This methodology is particularly important in common applications in chemical and pharmaceutical industries, where a good catalyst might cause losses of material, energy, and time if it is not effective. Decision-makers can assess catalysts through this framework by multidimensional perspectives of degrees of truth (effectiveness), indeterminacy (experimental variability), and falsity (instability or toxicity), leading to more well-informed, comprehensive, and realistic decisions.

Perhaps, the methodology will facilitate the ranking of numerous candidates in terms of score functions which normalize and assign different weights to various aspects of uncertainty in situations where classical data analysis would fail.

Consequently, the described neutrosophic over soft topology approach is mathematically sound and practically relevant, given the rise in interest of intelligent uncertain models in green chemistry and process optimization. By aiding researchers to select catalysts that are effective, safe, and sustainable, the proposed methodology will favour the advancement of cleaner technologies; knowledge-driven eco-conscious chemical manufacturing would further gain from this.

Consider the hydrogenation of alkenes, a fundamental organic reaction where alkenes are converted into alkanes using a metal catalyst and hydrogen gas. The general reaction is:
$R - CH = CH - R + H2-> [Catalyst]R - CH2 - CH2 - R$

The choice of catalyst significantly influences the rate, yield, and sustainability of the reaction.
Let $\mathscr{H}$ = {h_1, h_2, h_3, h_4} be a set of catalysts: • h1: Nickel (Ni)
h_2: Platinum (Pt)
h_3: Palladium (Pd)
h_4: Copper (Cu)

Define a neutrosophic over soft set on as:
$\langle h_1, 1.2, 0.5, 0.3\rangle$,
$\mathscr{J} = \langle h_2, 0.9, 1.3, 0.6\rangle$,
$\langle h_3, 0.8, 0.4, 1.1\rangle$,
$\langle h_4, 1.1, 0.7, 0.9\rangle$

Each triple $\{\langle \mathrm{h}, \aleph\mathscr{J}(\mathrm{h}), ð\mathscr{J}(\mathrm{h}), \Upsilon\mathscr{J}(\mathrm{h})\rangle\}$ denotes the degrees of:

Truth-membership (effectiveness/yield)
Indeterminacy-membership (experimental uncertainty)
Falsity-membership (side effects/cost/instability)

Now define a neutrosophic over soft topology on :
$\tau\mathcal{N}\mathfrak{so} = \{\triangleright, \blacktriangleright, \mathcal{U}_1, \mathcal{U}_2\}$

Where:
$\mathcal{U}_1 = \{\langle \mathrm{h}_1, 1.2, 0.5, 0.3\rangle, \langle \mathrm{h}_2, 0.9, 1.3, 0.6\rangle\}$, $\mathcal{U}_2 = \{\langle \mathrm{h}_3, 0.8, 0.4, 1.1\rangle, \langle \mathrm{h}_4, 1.1, 0.7, 0.9\rangle\}$

Then $(\mathscr{H}, \tau\mathcal{N}_{\mathfrak{s}}{}^{\mathfrak{o}})$ forms a neutrosophic over soft topological space.

Normalized Score Function for Catalyst Ranking
To prioritize the best catalyst, define a score function:

Raw Score:
$S_{\mathrm{raw}}(\mathrm{h}) = \aleph\mathscr{J}(\mathrm{h}) - \eth\mathscr{J}(\mathrm{h}) - \Upsilon\mathscr{J}(\mathrm{h})$
Raw Scores:
$S_{\mathrm{raw}}(\mathrm{h}_1) = 1.2 - 0.5 - 0.3 = 0.4$
$S_{\mathrm{raw}}(\mathrm{h}_2) = 0.9 - 1.3 - 0.6 = -1.0$
$S_{\mathrm{raw}}(\mathrm{h}_3) = 0.8 - 0.4 - 1.1 = -0.7$
$S_{\mathrm{raw}}(\mathrm{h}_4) = 1.1 - 0.7 - 0.9 = -0.5$

Normalization

$$S_{norm}(h) = \frac{s_{raw}(h) - s_{\min}}{s_{\max} - s_{\min}}$$

Where Smax = 0.4, Smin = –1.0

Ranking Based on Normalized Scores

This line graph0 illustrates the ranking of four elements based on their corresponding scores. Among them, h_1 achieves the highest score of 1.000, followed by h_4 with 0.357, h_3 with 0.214, and h_2 with the lowest score of 0.000. The graph presents these values in descending order using distinct markers and labels for clarity. This visual representation helps in easily identifying the most optimal element among the given set.

DISCUSSION

The results indicate that h_1 is the most optimal element based on the normalized scores, followed by h_4, h_3 and h_2. This ranking aligns well with the expected performance hierarchy of the elements under the given criteria. The clear distinction in scores supports the effectiveness of the proposed neutrosophic over soft topological approach for reliable decision-making in uncertain environments.

$$S_{\mathrm{norm}}(\mathrm{h}_1) = \frac{0.4-(-1.0)}{1.4} = \frac{1.4}{1.4} = 1.000$$

$$S_{\mathrm{norm}}(\mathrm{h}_2) = \frac{-1.0-(-1.0)}{1.4} = 0.000$$

$$S_{\mathrm{norm}}(\mathrm{h}_3) = \frac{-0.7-(-1.0)}{1.4} = \frac{0.3}{1.4} \approx 0.214$$

$$S_{\mathrm{norm}}(\mathrm{h}_4) = \frac{-0.5-(-1.0)}{1.4} = \frac{0.5}{1.4} \approx 0.357$$

CONCLUSION

An interesting advancement in the framework of Neutrosophic Over Soft Topological Spaces or $\mathcal{N}_s$-paces delineated in the below work is the introduction and investigation of different generalized open sets such as $\mathcal{N}_s^o\alpha$-, $\mathcal{N}_s^o\beta$-, $\mathcal{N}_s^o$ semi-, and $\mathcal{N}_s^o$ pre-open sets. Together with these, the corresponding types of continuity functions have also been defined, backed up by some propositions explaining their links. This theoretical insight into the new types of open sets enriches the already existing topological structure under $\mathcal{N}_s^o$ with potential development of finer classification and mapping schemes in uncertain situations. Moreover, various numerical instances like that of optimal catalyst selection for organic reactions are included in order to show practical usefulness of the newly proposed concepts, thus demonstrating relevance for application and real-life decision-making within the extended framework obtained.

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Conflict of interest

The authors declare no conflict of interest.

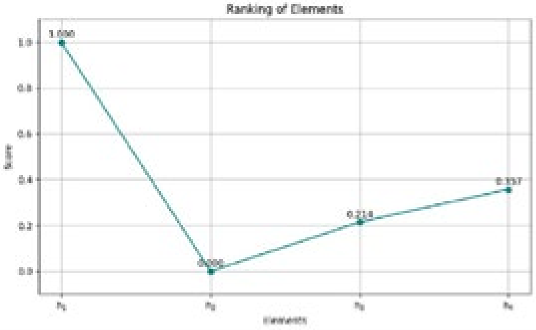

Fig. 1: Graphical Representation of Ranking of Elements

REFERENCES

1. Atanassov, K. T., Shannon, A. G., A note on intuitionistic fuzzy logics, *Acta Philosophica*, **1998**, *7(1),* 121–125.
2. Atanassov, K. T., Intuitionistic fuzzy modal topological structure, *Mathematics*, **2022**, *10(18),* 3313.
3. Bellman, R. E., Zadeh, L. A., Decision-making in a fuzzy environment, *Management Science*, **1970**, *17(4),* B141.
4. Broumi, S., Generalized neutrosophic soft set, Infinite Study, 2013.
5. Bustince, H., Burillo, P., Correlation of interval-valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems,* **1995**, *74(2),* 237–244.
6. Christianto, V., Boyd, R. N., Smarandache, F., Three possible applications of neutrosophic logic in fundamental and applied sciences, Infinite Study, 2020.
7. Devi, R. N., Parthiban, Y., Decision making by neutrosophic over soft topological space, *Neutrosophic Sets and Systems,* **2024**, *66,* 76–94.
8. Devi, R. N., Parthiban, Y., Exploring neutrosophic over supra exterior modal topological structure: Theory and application in healthcare decision-making, *Neutrosophic Sets and Systems,* **2024**, *72*, 272–285.
9. Devi, R. N., Parthiban, Y., Analyzing pharmaceutical compounds through neutrosophic over soft hyperconnected spaces, *Orient J Chem,* **2024**, *40(6),* 1627–1633.
10. Dhavaseelan, R., Devi, R. N., Jafari, S., Imran, Q. H., Neutrosophic alpha m-continuity, Infinite Study, 2019.
11. Jansi, R., Mohana, K., Smarandache, F., Correlation measure for pythagorean neutrosophic sets with T and F as dependent neutrosophic components, Infinite Study, 2019.
12. Kumaravel, S. K., Murugesan, R., Nagaram, N. B., Rasappan, S., Yamini, G., Applications of fuzzy cognitive maps and neutrosophic cognitive maps on analysis of dengue fever, *Fuzzy Logic Applications in Computer Science and Mathematics,* **2023**, 249–265.
13. Madhumathi, T., NirmalaIrudayam, F., Neutrosophic Orbit Topological Spaces, Trends in Sciences, **2021**, *18(24),* 1443–1443.
14. Madhumathi, T., Irudayam, F. N., Neutrosophic Orbit Continuous Mappings, *Neutrosophic Sets and Systems,* **2022***, 50*, 287.
15. Maji, P. K., Biswas, R., Roy, A. R., Soft set theory, *Computers & Mathematics with Applications,* **2003**, *45(4–5*), 555–562.
16. Mallick, R., Pramanik, S., Pentapartitioned neutrosophic set and its properties, *Infinite Study*, **2020**, *36*.
17. Mehmood, A., Nadeem, F., Park, C., Nordo, G., Kalsoom, H., Rahim Khan, M., Abbas, N., Neutrosophic Soft -Open Set in Neutrosophic Soft Topological Spaces, *Journal of Algorithms and Computation,* **2020**, *52(1),* 37–63.
18. Molodtsov, D., Soft set theory? First results, Computers & Mathematics with Applications, **1999**, *37(4–5),* 19–31.
19. Murugesan, R., Parthiban, Y., Devi, R., Mohan, K. R., Kumaravel, S. K., A comparative study of fuzzy cognitive maps and neutrosophic cognitive maps on COVID variants, Neutrosophic Sets and Systems, **2023**, *55(1),* 20.
20. Radha, R., Mary, A. S. A., Prema, R., Broumi, S., Neutrosophic pythagorean sets with dependent neutrosophic pythagorean components and its improved correlation coefficients*, Neutrosophic Sets and Systems,* **2021**, *46*, 77–86.
21. Majeed, R. N., El-Sheikh, S. A., *Fuzzy orbit topological spaces, Materials Science and Engineering,* 2019, *571*, 012026.
22. Saeed, M., Saqlain, M., Mehmood, A., Yaqoob, S., Multi-polar neutrosophic soft sets with application in medical diagnosis and decision-making, *Neutrosophic Sets and Systems,* **2020**, *33*, 183–207.
23. Smarandache, F., A unifying field in logics: Neutrosophic logic, *American Research* Press, 1999.
24. Smarandache, F., Pramanik, S., New trends in neutrosophic theory and applications, Infinite Study, **2016**, *1*.

25. Smarandache, F., Neutrosophic overset, neutrosophic underset, and neutrosophic offset. Similarly for neutrosophic over-/under-/off-logic, probability, and statistics, Infinite Study, 2016.
26. Wang, H., Smarandache, F., Zhang, Y., Sunderraman, R., Single valued neutrosophic sets, *Infinite Study,* **2010**, 12.
27. Ye, J., Another form of correlation coefficient between single valued neutrosophic sets and its multiple attribute decision-making method, *Neutrosophic Sets and Systems,* **2013**, *1(1),* 8–12.
28. Zadeh, L. A., Fuzzy sets, Information and Control, 1965, 8(3), 338–353.
29. Zadeh, L. A., Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems, 1978, 1(1), 3–28.